

Higher order fermionic and bosonic operators on cylinders and Hopf manifolds

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Abstract

Higher order higher spin operators are generalizations of k th-powers of the Dirac operator. In this paper, we study higher order higher spin operators defined on some conformally flat manifolds, namely cylinders and Hopf manifolds. We will also construct the kernels of these operators on these manifolds.

Keywords: Fermionic and bosonic operators, conformally flat manifolds, Kleinian group, fundamental solutions.

1 Introduction

The Rarita-Schwinger operators, as the generalizations of Dirac operators in the higher spin theory, have been studied in many papers in different spaces: for instance, in Euclidean space [2, 4, 9, 13], on spheres and real projective spaces [12], and on cylinders and some other conformally flat manifolds [10, 11]. The Maxwell operator and the higher spin Laplace operators are the generalizations of the Laplace operator in the higher spin theory in Euclidean space [1, 5, 14]. Higher order higher spin operators are first studied on the sphere in [15]. Then in [3], higher order higher spin operators are introduced for spin 1 and spin $\frac{3}{2}$ in Euclidean space. However, such operators have not been studied on other conformally flat manifolds, such as cylinders and Hopf manifolds.

Conformally flat manifolds are manifolds with atlases whose transition functions are Möbius transformations. They can be constructed by factoring out a subdomain U of

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either the sphere \mathbb{S}^m or \mathbb{R}^m by a Kleinian group Γ of the Möbius group, where Γ acts strongly discontinuously on U and Γ is not cyclic. This gives rise to a conformally flat manifold $U \setminus \Gamma$. Cylinders are examples of conformally flat manifolds of type $\mathbb{R}^m \setminus \mathbb{Z}^l$ where \mathbb{Z}^l is an integer lattice and $1 \leq l \leq m$.

In this paper, we define k th order higher spin operators on cylinders and Hopf manifolds, where k is a positive integer. Their fundamental solutions are also constructed by applying translation groups or dilation groups.

R. S. Krausshar and J. Ryan [7, 8] introduced Clifford analysis on cylinders and tori, making use of the fact that the universal covering space of all of these manifolds is \mathbb{R}^m . Then, provided the functions and kernels are l -periodic for some $l \in \{1, 2, \dots, m\}$, they used the projection map to obtain the equivalent function or kernel on those manifolds. J. Li, J. Ryan and J. Vanegas [10, 11] used similar ideas to construct Rarita-Schwinger operators on cylinders and some other conformally flat manifolds. We follow them by taking the kernel of the higher order higher spin operators in \mathbb{R}^m and using the translation group to construct new kernels that are l -fold periodic, which are then projected onto cylinders. In order to prove that the new kernels are well defined, we adapt the Eisenstein series argument developed by Krausshar [6].

2 Preliminaries

2.1 Clifford algebra

A real Clifford algebra Cl_m can be generated from \mathbb{R}^m by considering the relationship

$$\underline{x}^2 = -\|\underline{x}\|^2$$

for each $\underline{x} \in \mathbb{R}^m$. We have $\mathbb{R}^m \subseteq Cl_m$. If $\{e_1, \dots, e_m\}$ is an orthonormal basis for \mathbb{R}^m , then $\underline{x}^2 = -\|\underline{x}\|^2$ tells us that

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} is the Kronecker delta function. An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$, where $A = \{j_1, \dots, j_r\} \subset \{1, 2, \dots, m\}$ and $1 \leq j_1 < j_2 < \dots < j_r \leq m$. Hence for any element $a \in Cl_m$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$.

The Dirac operator in \mathbb{R}^m is defined to be

$$D_x := \sum_{i=1}^m e_i \partial_{x_i}.$$

Note $D_x^2 = -\Delta_x$, where Δ_x is the Laplacian in \mathbb{R}^m . A Cl_m -valued function $f(x)$ defined on a domain U in \mathbb{R}^m is left monogenic if $D_x f(x) = 0$. Since multiplication of Clifford

numbers is not commutative in general, there is a similar definition for right monogenic functions.

3 The higher order higher spin operator on cylinders

For an integer l , $1 \leq l \leq m$, we define the l -cylinder C_l to be the m -dimensional manifold $\mathbb{R}^m/\mathbb{Z}^l$, where \mathbb{Z}^l denote the l -dimensional lattice defined by $\mathbb{Z}^l := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_l$. We denote its members $m_1e_1 + \cdots + m_le_l$ for each $m_1, \dots, m_l \in \mathbb{Z}$ by a bold letter \mathbf{m} . When $l = m$, C_l is the m -torus, T_m . For each l the space \mathbb{R}^m is the universal covering group of the cylinder C_l . Hence there is a projection map $\pi_l : \mathbb{R}^m \longrightarrow C_l$.

An open subset U of the space \mathbb{R}^m is called l -fold periodic if for each $x \in U$ the point $x + \mathbf{m} \in U$. So $U' := \pi_l(U)$ is an open subset of the l -cylinder C_l .

Suppose that $U \in \mathbb{R}^m$ is a l -fold periodic open set. Let $f(x, u)$ be a function defined on $U \times \mathbb{R}^m$ with values in \mathcal{Cl}_m . Then we say that $f(x, u)$ is a l -fold periodic function if for each $x \in U$ we have that $f(x, u) = f(x + \mathbf{m}, u)$. Moreover, we will assume throughout that f is a monogenic polynomial homogeneous of degree j in u .

Now, if $f : U \times \mathbb{R}^m \longrightarrow \mathcal{Cl}_m$ is an l -fold periodic function, then the projection π_l induces a well defined function

$$f' : U' \times \mathbb{R}^m \longrightarrow \mathcal{Cl}_m,$$

where $f'(x', u) = f(x, u)$ for each $x' \in U'$ and x an arbitrary representative of $\pi_l^{-1}(x')$. Moreover, any function $f' : U' \times \mathbb{R}^m \longrightarrow \mathcal{Cl}_m$ lifts to an l -fold periodic function $f : U \times \mathbb{R}^m \longrightarrow \mathcal{Cl}_m$, where $U = \pi_l^{-1}(U')$.

The projection map π_l induces a projection of the higher order higher spin operator $\mathcal{D}_{1,k}$ to an operator $\mathcal{D}_{1,k}^{C_l}$ acting on domains on $C_l \times \mathbb{R}^m$ which is defined by replacing D_x and Δ_x with D'_x and Δ'_x in $\mathcal{D}_{1,k}$, respectively, where D'_x is the projection of the Dirac operator D_x and Δ'_x is the projection of the Laplace operator Δ_x . Essentially this is the method in [10]. That is

$$\begin{aligned} \mathcal{D}_{1,2n}^{C_l} &= \Delta_x^m - \frac{4n}{m+2n-2} \langle u, D'_x \rangle \langle D_u, D'_x \rangle \Delta_x^{m-1}; \\ \mathcal{D}_{1,2n-1}^{C_l} &= D'_x \Delta_x^{m-1} - \frac{2}{m+2n-2} u \langle D_u, D'_x \rangle \Delta_x^{m-1} - \frac{4n-4}{m+2n-2} \langle u, D'_x \rangle \langle D_u, D'_x \rangle \Delta_x^{m-2} D'_x. \end{aligned}$$

We call the operator $\mathcal{D}_{1,2n}^{C_l}$ an l -cylindrical higher order bosonic operator of spin 1 and $\mathcal{D}_{1,2n-1}^{C_l}$ an l -cylindrical higher order fermionic operator of spin $\frac{3}{2}$.

Fundamental solutions of $\mathcal{D}_{1,k}^{C_l}$

We follow the techniques in [10] and [3], requiring that the order of the operator $k < m$ when the dimension m is even. Let U a domain in \mathbb{R}^m . We recall the fundamental solution of the higher order higher spin operators $\mathcal{D}_{1,k}$ in \mathbb{R}^m :

$$E_{1,k}(x, u, v) = c_1 G_k(x) Z_1\left(\frac{xu}{\|x\|^2}, v\right),$$

where c_1 is a non-zero real constant, $Z_1(u, v)$ is the reproducing kernel of degree-1 homogeneous harmonic (respectively monogenic) polynomials if k is even (resp. odd), and

$$G_k(x) := \begin{cases} \frac{1}{\|x\|^{m-2n}}, & \text{if } k = 2n; \\ \frac{x}{\|x\|^{m-2n+2}}, & \text{if } k = 2n - 1. \end{cases}$$

Now we consider sums of the following form:

$$cot_{l,1,k}(x, u, v) = \sum_{(m_1, \dots, m_l) \in \mathbb{Z}^l} E_{1,k}(x + m_1 e_1 + \dots + m_l e_l, u, v), \quad (1)$$

for all $1 \leq p \leq m - k - 1$.

We will show these functions are defined on the l -fold periodic domain $\mathbb{R}^m / \mathbb{Z}^l$ for fixed u and v in \mathbb{R}^m and are $\mathcal{C}l_m$ -valued. They are also l -fold periodic functions.

To prove the locally uniform convergence of the series (1), use the locally normal convergence of the series

$$\sum_{\mathbf{m} \in \mathbb{Z}^l} G_k(x + \mathbf{m})$$

established by the following proposition [6].

Proposition 1. *Let $p \in \mathbb{N}$ with $1 \leq p \leq m - k - 1$. Let \mathbb{Z}^p be the p -dimensional lattice. Then the series*

$$\sum_{\mathbf{m} \in \mathbb{Z}^p} q_0^k(x + \mathbf{m})$$

converges normally in $\mathbb{R}^m \setminus \mathbb{Z}^p$.

Here q_0^k is in the kernel of the k th-power of D_x and q_0^k is exactly our G_k above. The proof follows from Proposition 2.2 appearing in [6].

Returning to the series defined by (1),

$$\begin{aligned} \cot_{l,1,k}(x, u, v) &= \sum_{(m_1, \dots, m_l) \in \mathbb{Z}^l} E_{1,k}(x + m_1 e_1 + \dots + m_l e_l, u, v) \\ &= \sum_{\mathbf{m} \in \mathbb{Z}^l} G_k(x + \mathbf{m}) Z_1\left(\frac{(x + \mathbf{m})u(x + \mathbf{m})}{\|x + \mathbf{m}\|^2}, v\right), \quad 1 \leq l \leq n - k - 1, \end{aligned}$$

we observe that $Z_1\left(\frac{(x + \mathbf{m})u(x + \mathbf{m})}{\|x + \mathbf{m}\|^2}, v\right)$ is a bounded function on a bounded domain in \mathbb{R}^m because its first variable, $\frac{(x + \mathbf{m})u(x + \mathbf{m})}{\|x + \mathbf{m}\|^2}$, is a reflection in the direction $\frac{(x + \mathbf{m})}{\|x + \mathbf{m}\|}$ for each \mathbf{m} , and hence is a linear transformation which is a continuous function. Furthermore, the norm of $\frac{(x + \mathbf{m})u(x + \mathbf{m})}{\|x + \mathbf{m}\|^2}$ is equal to the norm of u , so the bound of Z_1 with respect to the first variable does not depend on \mathbf{m} . On the other hand, we would get with respect to the second variable, bounded homogeneous functions of degree 1.

Consequently, applying the former proposition, the series

$$\cot_{l,1,k}(x, u, v) = \sum_{(m_1, \dots, m_l) \in \mathbb{Z}^l} E_{1,k}(x + m_1 e_1 + \dots + m_l e_l, u, v),$$

for $1 \leq l \leq m - k - 1$, is a uniformly convergent series and represents a kernel for the higher order higher spin operators under translations by $\mathbf{m} \in \mathbb{Z}^l$, with $1 \leq l \leq m - k - 1$.

Using similar argument in [10], we define the $(m - k)$ -fold periodic cotangent. In order to do that, we decompose the lattice \mathbb{Z}^l into three parts: the origin $\{0\}$ and a positive and a negative part. The last two parts are equal and disjoint:

$$\begin{aligned} \Lambda_l &= \{m_1 e_1 : m_1 \in \mathbb{N}\} \cup \{m_1 e_1 + m_2 e_2 : m_1, m_2 \in \mathbb{Z}, m_2 > 0\} \\ &\cup \dots \cup \{m_1 e_1 + \dots + m_l e_l : m_1, \dots, m_l \in \mathbb{Z}, m_l > 0\} \end{aligned}$$

and

$$-\Lambda_l = (\mathbb{Z}^l \setminus \{0\}) \setminus \Lambda_l.$$

For $l = m - k$, we define

$$\cot_{m-k,1,k}(x, u, v) = E_{1,k}(x, u, v) + \sum_{\mathbf{m} \in \Lambda_{m-k}} [E_{1,k}(x + \mathbf{m}, u, v) + E_{1,k}(x - \mathbf{m}, u, v)].$$

The following proposition [6] and a similar argument as in [10] shows the above series converges normally.

Proposition 2. *Let \mathbb{Z}^{m-k} be the $(m - k)$ -dimensional lattice. Then the series*

$$q_0^k(x) + \sum_{\mathbf{m} \in \mathbb{Z}^{m-k} \setminus \{0\}} (q_0^k(x + \mathbf{m}) - q_0^k(\mathbf{m}))$$

The proof follows Proposition 2.2 appearing in [6]. Hence, it is a kernel for the higher order higher spin operator under translation by $\mathbf{m} \in \Lambda_{m-k}$.

For $x, y \in \mathbb{R}^m$, the function $cot_{l,1,k}(x - y, u, v)$ induces functions on C_l :

$$cot'_{l,1,k}(x', y', u, v) = cot_{l,1,k}(x - y, u, v).$$

for each $x', y' \in U'$ and x, y arbitrary representatives of $\pi_l^{-1}(x')$ and $\pi_l^{-1}(y')$. These functions are defined on $(C_l \times C_l) \setminus diagonal(C_l \times C_l)$ for each fixed $u, v \in \mathbb{R}^m$, where

$$diagonal(C_l \times C_l) = \{(x', x') : x' \in C_l\}$$

and they satisfy

$$\mathcal{D}_{1,k}^{C_l} cot'_{l,1,k}(x', y', u, v) = 0.$$

Conformally inequivalent spinor bundles on C_l

Previously the spinor bundle over C_l was trivial, $C_l \times \mathcal{C}l_m$. However, there are 2^l spinor bundles on C_l . See more details in [11]. The following construction is for some of the spinor bundles over C_l and all the others can be constructed similarly.

First let p be an integer in the set $\{1, 2, \dots, l\}$ and consider the lattice $\mathbb{Z}^p := \mathbb{Z}e_1 + \dots + \mathbb{Z}e_p$. We also consider the lattice $\mathbb{Z}^{l-p} := \mathbb{Z}e_{p+1} + \dots + \mathbb{Z}e_l$. In this case $\mathbb{Z}^l = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \mathbb{Z}^p, \mathbf{n} \in \mathbb{Z}^{l-p}\}$. Suppose that $\mathbf{m} = m_1e_1 + \dots + m_pe_p$. Let us make the identification (x, X) with $(x + \mathbf{m} + \mathbf{n}, (-1)^{m_1 + \dots + m_p} X)$ where $x \in \mathbb{R}^m$ and $X \in \mathcal{C}l_m$. This identification gives rise to a spinor bundle E^p over C_l .

We adapt functions in previous section as follows. For $1 \leq l \leq m - k - 1$ we define

$$cot_{l,1,k,p}(x, u, v) = \sum_{\mathbf{m} \in \mathbb{Z}^p, \mathbf{n} \in \mathbb{Z}^{l-p}} (-1)^{m_1 + \dots + m_p} E_{1,k}(x + \mathbf{m} + \mathbf{n}, u, v).$$

These are well defined functions on $\mathbb{R}^m \setminus \mathbb{Z}^l$. Therefore, we obtain from these functions the cotangent kernels

$$cot_{l,1,k,p}(x, y, u, v) = \sum_{\mathbf{m} \in \mathbb{Z}^p, \mathbf{n} \in \mathbb{Z}^{l-p}} (-1)^{m_1 + \dots + m_p} E_{1,k}(x - y + \mathbf{m} + \mathbf{n}, u, v).$$

Again applying the projection map π_l these kernels give rise to the kernels

$$cot'_{l,1,k,p}(x', y', u, v).$$

In the case $l = m - k$, by considering the series

$$cot_{m-k,1,k}(x, y, u, v) = E_{1,k}(x, u, v) + \sum_{\mathbf{m} \in \Lambda_{m-k}} [E_{1,k}(x + \mathbf{m}, u, v) + E_{1,k}(x - \mathbf{m}, u, v)],$$

we obtain the kernel

$$\begin{aligned} & \cot_{m-k,1,k}(x, y, u, v) \\ = & E_{1,k}(x - y, u, v) + \sum_{\mathbf{m} \in \Lambda_{m-k}} [E_{1,k}(x - y + \mathbf{m}, u, v) + E_{1,k}(x - y - \mathbf{m}, u, v)], \end{aligned}$$

which in turn using the projection map induces kernels

$$\cot'_{m-k,1,k}(x', y', u, v).$$

Defining

$$\begin{aligned} & \cot_{m-k,1,k,p}(x, u, v) = E_{1,k}(x + \mathbf{m} + \mathbf{n}, u, v) \\ & + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^p, \mathbf{n} \in \mathbb{Z}^{m-k-p} \\ \mathbf{m} + \mathbf{n} \in \Lambda_{m-k}}} (-1)^{m_1 + \dots + m_p} [E_{1,k}(x + \mathbf{m} + \mathbf{n}, u, v) + E_{1,k}(x - \mathbf{m} - \mathbf{n}, u, v)], \end{aligned}$$

we obtain the cotangent kernels

$$\begin{aligned} & \cot_{m-k,1,k,p}(x, y, u, v) = E_{1,k}(x - y + \mathbf{m} + \mathbf{n}, u, v) \\ & + \sum_{\substack{\mathbf{m} \in \mathbb{Z}^p, \mathbf{n} \in \mathbb{Z}^{m-k-p} \\ \mathbf{m} + \mathbf{n} \in \Lambda_{m-k}}} (-1)^{m_1 + \dots + m_p} [E_{1,k}(x - y + \mathbf{m} + \mathbf{n}, u, v) + E_{1,k}(x - y - \mathbf{m} - \mathbf{n}, u, v)], \end{aligned}$$

and by π_l the kernels

$$\cot'_{m-k,1,k,p}(x', y', u, v).$$

4 The higher order higher spin operator on Hopf manifolds

In this section, we use similar arguments as in [11] and [3], also requiring that the order of the operator satisfies $k < m$ when the dimension m is even. It is worth pointing out that the two spin structures introduced there also apply for our fermionic and bosonic operators. Let $U = \mathbb{R}^m$ and $\Gamma = \{t^i : i \in \mathbb{Z}\}$, where t is an arbitrary strictly positive real number distinct from 1. Then by factoring out U by Γ we have the conformally flat spin manifold $\mathbb{S}^1 \times \mathbb{S}^{m-1}$ which we denote by H_m and call a Hopf manifold. As $\prod_1(\mathbb{S}^1 \times \mathbb{S}^{m-1}) = \mathbb{Z}$ for $m > 2$, it follows that the Hopf manifold H_m has two distinct spin structures. On this space, we can define the higher order higher spin operators and construct their kernels over the two different spinor bundles over $\mathbb{S}^1 \times \mathbb{S}^{m-1}$.

In all that follows $\mathbb{R}^m \setminus \{0\}$ will be a universal covering space of the conformally flat manifold $\mathbb{S}^1 \times \mathbb{S}^{m-1}$ ([8]). So there is a projection map $p : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^{m-1}$. Further for each $x \in \mathbb{R}^m \setminus \{0\}$ we shall denote $p(x)$ by x' . Further if V is a subset of $\mathbb{R}^m \setminus \{0\}$ then

we denote $p(V)$ by V' .

One spinor bundle F_1 over $\mathbb{S}^1 \times \mathbb{S}^{m-1}$ can be constructed by identifying the pair (x, X) with $(t^i x, t^{\frac{i(m-1)}{2}} X)$ for every $k \in \mathbb{Z}$, where $x \in \mathbb{R}^m \setminus \{0\}$ and $X \in \mathcal{C}l_m$.

Consider a domain $V \subset \mathbb{R}^m \setminus \{0\}$ satisfying $t^i x \in V$ for each $i \in \mathbb{Z}$ and $x \in V$. We will call a such domain a k -factor dilation domain. Further we define a i -factor dilation function as a function $f(x, u) : V \times \mathbb{R}^m \rightarrow \mathcal{C}l_m$ such that f is a monogenic polynomial homogeneous of degree l in u satisfying $f(x, u) = t^{\frac{i(m-1)}{2}} f(t^i x, u)$ for each $x \in V$ and each integer i .

The projection map p induces a well defined function

$$f' : V' \times \mathbb{R}^m \rightarrow F_1,$$

where $f'(x, u) = t^{\frac{i(m-1)}{2}} f(x, u)$ for each $x' \in V'$ and x an arbitrary representative of $\pi_l^{-1}(x')$.

The higher order higher spin operator over $\mathbb{R}^m \setminus \{0\}$ induces a higher order higher spin operator acting on sections of the bundle F_1 over H_m . We will denote this operator by $\mathcal{D}_{1,k}^{H_m}$. If $\mathcal{D}_{1,k}^{H_m}(f') = 0$ then f' is called an F_1 -left higher order higher spin section. Moreover, any F_1 -left higher order higher spin section $f' : V' \times \mathbb{R}^m \rightarrow F_1$ lifts to a k -factor dilation function $f : V \times \mathbb{R}^m \rightarrow \mathcal{C}l_m$, where $V = p^{-1}(V')$ and $\mathcal{D}_{1,k}(f) = 0$.

Now we consider the series

$$\begin{aligned} E_{1,k,1}^{H_m}(x, y, u, v) &= \sum_{i=-\infty}^0 G_k(t^i x - t^i y) Z_1\left(\frac{(t^i x - t^i y)u(t^i x - t^i y)}{\|t^i x - t^i y\|^2}, v\right) \\ &+ t^{2(k-m)} G_k(x) Z_1\left(\frac{xux}{\|x\|^2}, v\right) \left[\sum_{i=1}^{\infty} G_k(t^{-i} x^{-1} - t^{-i} y^{-1}) \right. \\ &\quad \left. Z_1\left(\frac{(t^{-i} x^{-1} - t^{-i} y^{-1})u(t^{-i} x^{-1} - t^{-i} y^{-1})}{\|t^{-i} x^{-1} - t^{-i} y^{-1}\|^2}, v\right) \right] G_k(y) Z_1\left(\frac{yuy}{\|y\|^2}, v\right), \end{aligned}$$

where $x, y \in \mathbb{R}^m \setminus \{0\}$ and $y \neq t^i x$ for all $i \in \mathbb{Z}$. From the definition of $Z_1(u, v)$ and the homogeneity of $G_k(x)$, it is easy to obtain that $E_{1,k,1}^{H_m}$ converges normally on any compact subset K not containing the points $y = t^i x$ where $i \in \mathbb{Z}$. Since $E_{1,k,1}^{H_m}(tx, ty, u, v) = E_{1,k,1}^{H_m}(x, y, u, v)$, this kernel is periodic with respect to the Kleinian group $\{t^i : i \in \mathbb{Z}\}$. The kernel for the higher order higher spin operators on $(\mathbb{S}^1 \times \mathbb{S}^{m-1}) \times (\mathbb{S}^1 \times \mathbb{S}^{m-1}) \setminus \text{diagonal}(\mathbb{S}^1 \times \mathbb{S}^{m-1})$ is then the projection of $E_{1,k,1}^{H_m}(x, y, u, v)$ on $(\mathbb{S}^1 \times \mathbb{S}^{m-1}) \times (\mathbb{S}^1 \times \mathbb{S}^{m-1}) \setminus \text{diagonal}(\mathbb{S}^1 \times \mathbb{S}^{m-1})$:

$$E_{1,k,1}^{H_m}(x', y', u, v) = E_{1,k,1}^{H_m}(x, y, u, v),$$

for x, y representatives of $\pi_l^{-1}(x')$ and $\pi_l^{-1}(y')$ as earlier.

The second spinor bundle F_2 over $\mathbb{S}^1 \times \mathbb{S}^{m-1}$ can be constructed by identifying the pair (x, X) with $(t^i x, (-1)^i t^{\frac{i(m-1)}{2}} X)$.

Now we introduce the normally convergent series:

$$\begin{aligned} E_{1,k,2}^{H_m}(x, y, u, v) &= \sum_{i=-\infty}^0 (-1)^i G_k(t^i x - t^i y) Z_1\left(\frac{(t^i x - t^i y)u(t^i x - t^i y)}{\|(t^i x - t^i y)\|^2}, v\right) \\ &+ (-1)^i t^{2(k-m)} G_k(x) Z_1\left(\frac{xux}{\|x\|^2}, v\right) \left[\sum_{i=1}^{\infty} G_k(t^{-i} x^{-1} - t^{-i} y^{-1}) \right. \\ &\quad \left. Z_1\left(\frac{(t^{-i} x^{-1} - t^{-i} y^{-1})u(t^{-i} x^{-1} - t^{-i} y^{-1})}{\|t^{-i} x^{-1} - t^{-i} y^{-1}\|^2}, v\right) \right] G_k(y) Z_1\left(\frac{yuy}{\|y\|^2}, v\right) \end{aligned}$$

where $x, y \in \mathbb{R}^m \setminus \{0\}$ and $y \neq t^i x$ for all $i \in \mathbb{Z}$. This function induces through the projection map p on the variable $x, y \in \mathbb{R}^m \setminus \{0\}$, the higher order higher spin kernel associated with F_2 :

$$E_{1,k,2}^{H_m}(x', y', u, v) = E_{1,k,2}^{H_m}(x, y, u, v),$$

for x and y again as earlier.

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